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Properties of locally linearly independent refinable function vectors

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Abstract

The paper considers properties of compactly supported, locally linearly independent refinable function vectors $\Phi = (\phi_1, \ldots, \phi_r)^T$, $r \in \mathbb{N}$. In the first part of the paper, we show that the interval endpoints of the global support of ϕ_ν, $\nu = 1, \ldots, r$, are special rational numbers. Moreover, in contrast with the scalar case $r = 1$, we show that components ϕ_ν of a locally linearly independent refinable function vector Φ can have holes. In the second part of the paper we investigate the problem whether any shift-invariant space generated by a refinable function vector Φ possesses a basis which is linearly independent over $(0, 1)$. We show that this is not the case. Hence the result of Jia, that each finitely generated shift-invariant space possesses a globally linearly independent basis, is in a certain sense the strongest result which can be obtained.

Keywords: Refinable function vectors; Local linear independence; Global linear independence; Support of refinable functions

1. Introduction

In this paper, we are especially interested in properties of refinable function vectors which are locally linearly independent.

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Let $\Phi = (\phi_1, \dots, \phi_r)^T$, $r \in \mathbb{N}$, $r \geqslant 1$, be a vector of compactly supported integrable functions on $\mathbb{R}$. A function vector Φ is said to be *refinable* if it satisfies a refinement equation

$$\Phi(t) = \sum_{k \in \mathbb{Z}} A(k)\Phi(2t-k), \quad t \in \mathbb{R}, \tag{1}$$

where $\{A(k)\}$ is a finitely supported sequence of $(r \times r)$-matrices.

We say that Φ is *linearly independent over a nonempty open subset* G of $\mathbb{R}$, if for any sequences $c_1, \dots, c_r$ on $\mathbb{Z}$,

$$\sum_{v=1}^{r} \sum_{k \in \mathbb{Z}} c_v(k)\phi_v(\cdot - k) = 0 \quad \text{on } G$$

implies that $c_v(k) = 0$ for all $k \in I_v(G)$, $v = 1, \dots, r$, where $I_v(G)$ contains all $k \in \mathbb{Z}$ with $\phi_v(\cdot - k) \not\equiv 0$ on G. Further, Φ is *locally linearly independent* (l. l. i.) if it is linearly independent over any nonempty open subset G of $\mathbb{R}$.

We say that Φ is *globally linearly independent* (g. l. i.) if, for any sequences $c_1, \dots, c_r$ on $\mathbb{Z}$,

$$\sum_{v=1}^{r} \sum_{k \in \mathbb{Z}} c_v(k)\phi_v(\cdot - k) = 0 \quad \text{on } \mathbb{R}$$

implies that $c_v(k) = 0$ for all $v = 1, \dots, r$ and all $k \in \mathbb{Z}$, see [13].

The concept of local linear independence has been intensively studied in spline approximation (see e.g. [4,11,23]). In wavelet analysis the notions of global and local linear independence have been used as a tool for wavelet approximation and for construction of wavelets on the interval (see e.g. [5,12,17,18]).

For $r = 1$, the refinement equation (1) is of the form

$$\phi(x) = \sum_{k=a}^{b} A(k)\phi(2t-k), \quad a, b \in \mathbb{Z}. \tag{2}$$

In this case, it was shown by Lemarié [17] that the global linear independence is equivalent to the local linear independence on the unit interval $(0, 1)$. Sun [25] stated that local and global linear independence are equivalent for a function ϕ satisfying (2).

Let the global support of an integrable function f, gsupp f, be the smallest interval $I \subset \mathbb{R}$ with supp $f \subseteq I$. Then for a l. l. i. function ϕ satisfying (2) it follows that supp ϕ = gsupp $\phi = [a, b]$ if $A(a), A(b) \neq 0$, i.e., ϕ has integer support. Moreover, ϕ cannot have a hole, i.e., there is no interval of Lebesgue measure greater than zero lying inside the global support of ϕ where ϕ vanishes. Further, the integer translates of a l. l. i. function ϕ satisfy the minimality property, i.e., for every compactly supported ψ being a linear combination of integer translates of ϕ it follows that gsupp $\psi \supseteq$ gsupp $\phi(\cdot - k)$ for some $k \in \mathbb{Z}$, and equality holds if and only if $\psi = c\phi(\cdot - k)$ for some constant $c \neq 0$ (see [3,21]).

For compactly supported refinable function vectors Φ, global and local linear independence are no longer equivalent (see [7]).

The two properties, local as well as global linear independence can be completely characterized by the matrix mask $\{A(k)\}$ of Φ, see e.g. [1,6,9,16,26] for the univariate vector case, [10] for multivariate scaling functions and [7] for the multivariate vector case.

In this paper, we study the support properties of l. l. i. function vectors. While for a single refinable l. l. i. function ϕ we have the above-mentioned useful properties, little is known for the vector case. Estimates and computations of the global support of refinable function vectors Φ have been given by Heil and Collela [8], Ruch et al. [22,24] and by Plonka [20].

We shall answer the following questions in the first part of the paper: What does the global support of the components for l. l. i. function vectors look like? Can components of a l. l. i. refinable function vector Φ have holes? Is a g. l. i. function vector also linearly independent over a finite interval?

In the second part of the paper, we study bases of shift-invariant spaces. As shown by Jia [12], any finitely generated shift-invariant space possesses a globally linearly independent basis (see Theorem B in Section 4). One can ask the question, whether this result can be strengthened in the following direction: Does any shift-invariant space generated by a refinable function vector Φ have a basis which is linearly independent over $(0,1)$? Unfortunately this is not the case. Hence, the result of Jia is in this sense the strongest result which can be obtained.

The paper is organized as follows. In Section 2, we briefly recall the characterization of local linear independence of Φ in terms of the mask. In Section 3, we study support properties of l. l. i. function vectors. In particular, we show that the global supports of the components ϕ_v, $v = 1, \dots, r$, of Φ start and end with special rational numbers. We present a compactly supported, continuous, refinable function vector, which is l. l. i. but has a component possessing a hole in its global support. We also show that, if Φ satisfies (1) with $A(k) = 0$ for $k<0$ and $k>N$, and if $A(0)$ and $A(N)$ do not contain zero rows, then the components of Φ have no holes.

Finally, in Section 4, we present an example of a refinable function vector which is g. l. i. but not linearly independent over $(0,1)$, and where the shift-invariant space generated by Φ does not possess a basis being linearly independent over (0,1). Moreover, we show that there are refinable function vectors being g. l. i. but linearly dependent over any finite interval.

2. Characterization of local linear independence

Let us briefly recall the characterization of local linear independence from Goodman et al. [7] and Cheung et al. [2].

We assume that the mask $\{A(k)\}$ is supported on $[0, N]$, i.e., for $k<0$ and $k>N$ the $(r \times r)$-matrices $A(k)$ are zero matrices. Let

$$\mathbf{\Phi}(t) := (\Phi(t+k))_{k=0}^{N-1} \quad \text{for } t \in [0,1).$$

Then, for each $t \in [0, 1)$, $\mathbf{\Phi}(t)$ is a vector of length rN. With the help of the two-slanted block matrices

$$\mathscr{A}_0 := (A(2k-l))_{k,l=0}^{N-1}, \quad \mathscr{A}_1 := (A(2k-l+1))_{k,l=0}^{N-1}$$

the refinement equation (1) implies

$$\mathbf{\Phi}\left(\frac{t}{2}\right) = \mathscr{A}_0 \mathbf{\Phi}(t) \quad \text{and} \quad \mathbf{\Phi}\left(\frac{t+1}{2}\right) = \mathscr{A}_1 \mathbf{\Phi}(t) \tag{3}$$

for $t \in [0, 1)$. It follows that for $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$, we have

$$\mathbf{\Phi}\left(\frac{\varepsilon_1}{2} + \cdots + \frac{\varepsilon_n}{2^n} + \frac{t}{2^n}\right) = \mathscr{A}_{\varepsilon_1} \ldots \mathscr{A}_{\varepsilon_n} \mathbf{\Phi}(t) \quad t \in [0, 1).$$

Suppose that $\Phi \in (L_1(\mathbb{R}))^r$ is a nontrivial compactly supported solution of (1) (with $A(k) = 0$ for $k < 0$ and $k > N$). Let

$$v_0 := \int_0^1 \mathbf{\Phi}(t)\, dt = \left(\int_0^1 \Phi(t+k)\, \mathrm{d}t\right)_{k=0}^{N-1} \in \mathbb{R}^{rN}.$$

Then v_0 is a right eigenvector of $\frac{1}{2}(\mathscr{A}_0 + \mathscr{A}_1)$ to the eigenvalue 1 ([2, Lemma 3.1]). Now, let V be the minimal common invariant subspace of $\{\mathscr{A}_0, \mathscr{A}_1\}$ generated by v_0. Further, let $\mathscr{B} = (\mathscr{B}(k,l))$ be an $(rN \times \dim V)$-matrix such that the columns of $\mathscr{B}$ form a basis of V. For continuous functions, instead of v_0 we can also choose a right eigenvector v of $\mathscr{A}_0$ to the eigenvalue 1 in order to generate the space V. In this case, V contains the vectors $\mathbf{\Phi}(t)$ with $t \in [0, 1)$, since for each t there is a sequence of dyadic numbers with the limit t. We have:

Theorem A. (Cheung et al. [2], Goodman et al. [7]). *Let Φ be a compactly supported, integrable solution vector of* (1) *with $A(k) = 0$ for $k < 0$ and $k > N$. Then we have*

(1) *Φ is linearly independent over $(0,1)$ if and only if the nonzero rows of $\mathscr{B}$ are linearly independent.*
(2) *Φ is locally linearly independent if and only if for all n with $0 \leqslant n \leqslant 2^{rN}$ and all $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ the nonzero rows of $\mathscr{A}_{\varepsilon_1} \ldots \mathscr{A}_{\varepsilon_n} \mathscr{B}$ are linearly independent.*

In [7], a procedure is presented which simplifies the application of Theorem A in order to investigate, if Φ is locally linearly independent or not.

3. Supports of locally linearly independent refinable vectors

As known, for l.l.i. refinable functions ϕ satisfying (2), it follows that $\operatorname{supp} \phi = [a,b]$, and in particular, ϕ has no holes (see [17]). Now we want to consider the support properties of l.l.i. function vectors in more detail.

First, the local linear independence implies the following restrictions on the starting point and endpoint of the global supports of the components ϕ_ν, $\nu = 1, \ldots, r$, of the refinable function vector Φ.

Theorem 1. *Let $\Phi = (\phi_1, \ldots, \phi_r)^T$, $r \in \mathbb{N}$, $r \geqslant 1$, be a refinable, locally linearly independent vector of compactly supported functions $\phi_\nu \in L^1(\mathbb{R})$. Then the starting point and the endpoint of* $\operatorname{gsupp} \phi_\nu$, $\nu = 1, \ldots, r$, *is a rational number of the form $k + c_r$, where $k \in \mathbb{Z}$ and $c_r \in J_r$ with*

$$J_r := \left\{ \frac{m}{(2^l - 1)\, 2^{r-l}} : l = 1, \ldots, r,\ m = 0, \ldots, (2^l - 1)\ 2^{r-l} - 1 \right\}.$$

In particular,

$$J_1 = \{0\}, \quad J_2 = \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\right\},$$

$$J_3 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}\right\}.$$

Proof. Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ with $\operatorname{gsupp} \phi_\nu = [a_\nu, b_\nu]$. We can assume that all starting points lie in $[0, 1)$, this is obtained by shifting the components of Φ without changing the local linear independence. Since the components of Φ are compactly supported and Φ is l.l.i. and refinable, the refinement mask of Φ is finite, i.e., there exist $a, b \in \mathbb{Z}$ with

$$\Phi(t) = \sum_{k=a}^{b} A(k) \Phi(2t - k), \quad t \in \mathbb{R}$$

with $(r \times r)$-matrices $A(k) = (A_{\mu,\nu}(k))_{\mu,\nu=1}^r$. Further, for $t \in \mathbb{R} \backslash [a_\mu,\ b_\mu]$ we have

$$\phi_\mu(t) = 0 = \sum_{k=a}^{b} \sum_{\nu=1}^{r} A_{\mu,\nu}(k)\ \phi_\nu(2t - k)$$

and the local linear independence of Φ implies that for all k with $A_{\mu,\nu}(k) \neq 0$,

$$\operatorname{gsupp} \phi_\nu(2 \cdot -k) \subseteq \operatorname{gsupp} \phi_\mu, \quad \mu, \nu = 1, \ldots, r.$$

Hence

$$\left[\frac{a_\nu}{2} + \frac{k}{2}, \frac{b_\nu}{2} + \frac{k}{2} \right] \subseteq [a_\mu, b_\mu],$$

such that the starting points (and endpoints) satisfy $k \geqslant 2a_\mu - a_\nu$ (and $k \leqslant 2b_\mu - b_\nu$) for all k with $A_{\mu,\nu}(k) \neq 0$. Moreover, for each fixed μ, one of the r inequalities for the starting points (and for the endpoints, respectively) must be an equality. Hence, for each fixed μ, there exists at least one $\nu \in \{1, \ldots, r\}$ with $2a_\mu - a_\nu \in \mathbb{Z}$ (and one $\tilde{\nu} \in \{1, \ldots, r\}$ with $2b_\mu - b_{\tilde{\nu}} \in \mathbb{Z}$).

We now consider the starting points more precisely. Since $0 \leqslant a_\nu < 1$ for $\nu = 1, \ldots, r$, we have (at least) r relations of the form

$$2a_\mu - a_{\nu(\mu)} \in \{0,\ 1\}, \quad \mu = 1, \ldots, r, \ \ \nu(\mu) \in \{1, \ldots, r\},$$

and we can find a cycle $\{\mu_1, \ldots, \mu_d\}, d \leqslant r$, such that

$$2a_{\mu_j} - a_{\mu_{j+1}} \in \{0, 1\}, \quad j = 1, \ldots, d-1, \quad 2a_{\mu_d} - a_{\mu_1} \in \{0, 1\}. \tag{4}$$

Considering the circulant $d \times d$ matrix

$$\operatorname{circ}(x_0, x_1, \ldots, x_{d-1}) := \begin{pmatrix} x_0 & x_{d-1} & x_{d-2} & \cdots & x_1 \\ x_1 & x_0 & x_{d-1} & \cdots & x_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ x_{d-2} & x_{d-3} & \ddots & \ddots & x_{d-1} \\ x_{d-1} & x_{d-2} & \cdots & x_1 & x_0 \end{pmatrix}$$

we obtain from (4) a system of linear equations

$$\operatorname{circ}(2, 0, \ldots, 0, -1)\mathbf{a} = \mathbf{e},$$

where $\mathbf{a} := (a_{\mu_1}, \ldots, a_{\mu_d})^T$ is the vector of starting points and $\mathbf{e}$ is an integer vector $(\delta_1, \ldots, \delta_d)^T$ with $\delta_j \in \{0, 1\}$. Hence, with

$$(\operatorname{circ}(2, 0, \ldots, 0, -1))^{-1} = \frac{1}{2^d - 1} \operatorname{circ}(2^{d-1}, 1, 2, \ldots, 2^{d-2})$$

we find

$$\mathbf{a} = \frac{1}{2^d - 1} \operatorname{circ}(2^{d-1}, 1, 2, \ldots, 2^{d-2})\mathbf{e}.$$

Observe that at least one component δ_v $(v = 1, \ldots, d)$ must be zero since $a_{\mu_1}, \ldots, a_{\mu_d} \in [0, 1)$. It follows that each a_{μ_j} must be a rational number of the form $\frac{m}{2^d - 1}$, $m \in \{0, \ldots, 2^d - 2\}$.

Further, for each $a_{\mu'}$ with μ' not belonging to a cycle, there exists a chain $\{\mu'_1, \ldots, \mu'_g\}$ with $\mu'_1 = \mu'$,

$$2a_{\mu'_j} - a_{\mu'_{j+1}} \in \{0, 1\} \quad \text{for } j = 1, \ldots, g-1$$

and μ'_g belongs to a cycle, but μ'_{g-1} does not. Hence, $a_{\mu'_g} = \frac{m}{2^d - 1} < 1$ for some $d < r$ and some $m \in \{0, \ldots, 2^d - 2\}$ and

$$a_{\mu'_j} = \frac{m'}{(2^d - 1)2^{g-j}} < 1$$

with some $m' \in \{0, \ldots, (2^d - 1)2^{g-j} - 1\}$, where m' depends on m and the number of equations of the form $2a_{\mu'_j} - a_{\mu'_{j+1}} = 1$ in the chain.

Observing that $d + g - 1 \leqslant r$, we find $a_{\mu'} = \frac{m'}{(2^d - 1)2^{g-1}} \in J_r$.

For the endpoints of the global support of ϕ_v, the proof follows analogously. □

While the support conditions in Theorem 1 are necessary consequences of the local linear independence of Φ, there may not exist refinable, l.l.i. function vectors with such exotic support intervals. However, for $r = 2$ we can show in the following examples, that indeed, all starting points and endpoints in J_2 can occur.

Example 1. Let $\Phi = (\phi_1, \phi_2)^T$ be a nonzero solution of the refinement equation

$$\Phi(t) = \begin{pmatrix} 0 & 0 \\ 4/5 & 3/5 \end{pmatrix} \Phi(2t) + \begin{pmatrix} 1/2 & 1/4 \\ 1/3 & 5/6 \end{pmatrix} \Phi(2t-1) + \begin{pmatrix} -1/5 & 3/5 \\ 0 & 0 \end{pmatrix} \Phi(2t-2).$$

Then Φ is continuous, locally linearly independent, and $\operatorname{supp} \phi_1 = [1/2, 2]$, $\operatorname{supp} \phi_2 = [0, 3/2]$, see Fig. 1.

Proof. We have

$$\mathscr{A}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4/5 & 3/5 & 0 & 0 \\ -1/5 & 3/5 & 1/2 & 1/4 \\ 0 & 0 & 1/3 & 5/6 \end{pmatrix}, \quad \mathscr{A}_1 = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 \\ 1/3 & 5/6 & 4/5 & 3/5 \\ 0 & 0 & -1/5 & 3/5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In order to show continuity of ϕ_1 and ϕ_2, we apply the following result of Jia et al. [15]. Let $\{A(k)\}_{k=0}^N$ be a finite refinement mask satisfying that $\frac{1}{2}\sum_{k=0}^N A(k)$ has one simple eigenvalue 1 and all other eigenvalues lie inside the unit circle. Then the subdivision scheme associated with A converges uniformly if and only if

(a) The mask $\{A(k)\}_{k=0}^N$ satisfies the sum rule of order 1, i.e., the matrices $\mathscr{A}_0$ and $\mathscr{A}_1$ both have the eigenvalue 1, and there exists a vector $e_1 \in \mathbb{R}^{rN}$ with $e_1^T \mathscr{A}_0 = e_1^T \mathscr{A}_1 = e_1^T$.

(b) Considering the subspace $U := \{u \in \mathbb{R}^{rN} : e_1^T u = 0\}$ the joint spectral radius of $\mathscr{A}_0|_U$ and $\mathscr{A}_1|_U$ satisfies

$$\rho(\mathscr{A}_0|_U, \mathscr{A}_1|_U) < 1.$$

Observe that the joint spectral radius is given by

$$\rho(\mathscr{A}_0|_U, \mathscr{A}_1|_U) = \inf_{n \geqslant 1} (\max\{\|\mathscr{A}_{\varepsilon_1}|_U \dots \mathscr{A}_{\varepsilon_n}|_U\| : \varepsilon_i \in \{0, 1\},\ i = 1, \dots, n\})^{1/n}$$

with an arbitrary matrix norm in $\mathbb{R}^{rN}$.

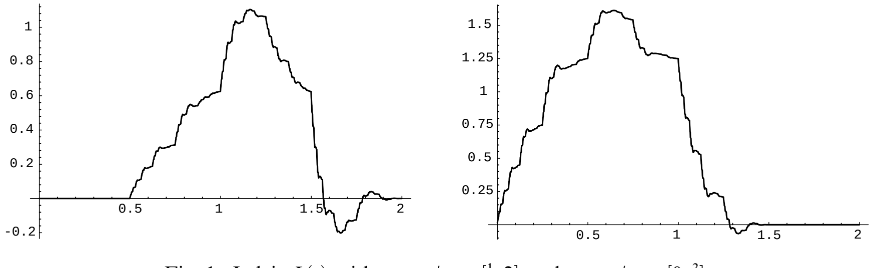

Fig. 1. L. l. i. $\Phi(t)$ with $\operatorname{supp} \phi_1 = [\frac{1}{2}, 2]$ and $\operatorname{supp} \phi_2 = [0, \frac{3}{2}]$.

In our example, the matrix $\frac{1}{2}\sum_{k=0}^{2} A(k)$ has the eigenvalues 1 and $-2/15$. Further, the mask satisfies the sum rule of order 1 with $e_1 = (2,3,2,3)^T$. Consider the subspace

$$U := \{u = (u_1, u_2, u_3, u_4)^T \in \mathbb{R}^4 : 2u_1 + 3u_2 + 2u_2 + 3u_4 = 0\}.$$

We choose a basis of U as $u_1 = (0, 64/5, 84/5, -24)^T$, $u_2 := (21, 0, 0, -14)^T$ and $u_3 := (-15, 42, -48, 0)^T$. Then the matrix representations of $\mathscr{A}_0|_U$ and $\mathscr{A}_1|_U$ under this basis are

$$\begin{pmatrix} 3/5 & 161/440 & 27/44 \\ 0 & 34/165 & 1/11 \\ 0 & 238/825 & 7/55 \end{pmatrix}, \quad \begin{pmatrix} -7/33 & -7/22 & 0 \\ 4/11 & 6/11 & 0 \\ 244/825 & 7/110 & -1/5 \end{pmatrix}.$$

The maximum column sum norms of these two matrices are less than 1, hence the joint spectral radius is less than 1. Thus, the subdivision scheme associated with this mask converges uniformly and the solution Φ is continuous.

Now we prove that Φ is l.l.i. The space V, as defined in Section 2, is spanned by the right eigenvector v_0 of $\frac{1}{2}(\mathscr{A}_0 + \mathscr{A}_1)$ to the eigenvalue 1, $v_0 = (3/2, 9, 7/2, 1)^T$, and $\mathscr{A}_1 v_0, \mathscr{A}_0^2 v_0, \mathscr{A}_1\mathscr{A}_0 v_0$, i.e., V has full dimension 4. Thus, by Theorem A, Φ is linearly independent over $(0,1)$ and the matrix B can be chosen as the (4×4)-identity matrix.

Now, we have rank $\mathscr{A}_0 =$ rank $\mathscr{A}_1 = 3$ and $\mathscr{A}_0$ has a zero row at the top and $\mathscr{A}_1$ has a zero row at the bottom. Further, $\mathscr{A}_0\mathscr{A}_0, \mathscr{A}_0\mathscr{A}_1, \mathscr{A}_1\mathscr{A}_0, \mathscr{A}_1\mathscr{A}_1$, all have rank 3 and $\mathscr{A}_0\mathscr{A}_0, \mathscr{A}_0\mathscr{A}_1$ have one zero row at the top, and $\mathscr{A}_1\mathscr{A}_0, \mathscr{A}_1\mathscr{A}_1$ a zero row at the bottom. Using the procedure proposed in [7], it already follows that Φ is l.l.i. Moreover, the structure of $A(0)$ and $A(2)$ implies that supp $\phi_1 = [\frac{1}{2}, 2]$ and supp $\phi_2 = [0, \frac{3}{2}]$. $\square$

Example 2. (cf. Goodman and Lee [6]). Consider $\Phi = (\phi_1, \phi_2)^T$ with

$$\Phi(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi(2t) + \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}\Phi(2t-1) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi(2t-2).$$

It can be simply observed that the piecewise linear splines

$$\phi_1(t) = \begin{cases} 3t-1 & t\in[1/3, 2/3) \\ -3t/2+2 & t\in[2/3, 4/3] \\ 0 & t\notin[1/3, 4/3] \end{cases}$$

$$\phi_2(t) = \begin{cases} 3t/2-1 & t\in[2/3, 4/3) \\ -3t+5 & t\in[4/3, 5/3] \\ 0 & t\notin[2/3, 5/3] \end{cases}$$

satisfy the above refinement equation, see Fig. 2.

We show that Φ is locally linearly independent.

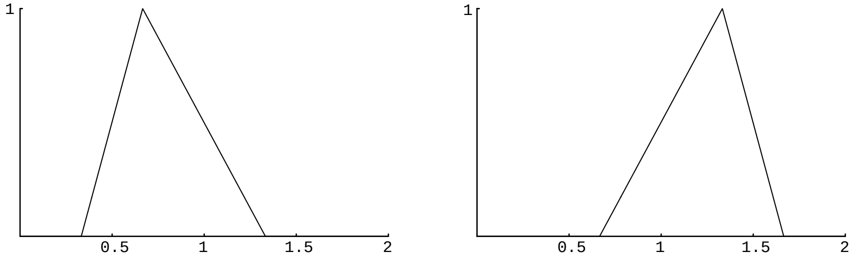

Fig. 2. L. l. i. $\Phi(t)$ with supp $\phi_1 = [1/3, 4/3]$ and supp $\phi_2 = [2/3, 5/3]$.

Consider

$$\mathscr{A}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 1 & 0 & 1/4 & 3/4 \end{pmatrix}, \quad \mathscr{A}_1 = \begin{pmatrix} 3/4 & 1/4 & 0 & 1 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then the space V has full dimension 4. A simple computation by Maple tells us that

rank $\mathscr{A}_0 =$ rank $\mathscr{A}_0\mathscr{A}_1 = 3$ and the 2nd rows are zero,

rank $\mathscr{A}_1 =$ rank $\mathscr{A}_1\mathscr{A}_0 = 3$ and the 3rd rows are zero,

rank $\mathscr{A}_1\mathscr{A}_0^2 = \mathscr{A}_0\mathscr{A}_1^2 = 2$ and the middle two rows are zero,

rank $\mathscr{A}_0^2 =$ rank $\mathscr{A}_0^3 =$ rank $\mathscr{A}_0\mathscr{A}_1\mathscr{A}_0^2 =$ rank $\mathscr{A}_0^2\mathscr{A}_1^2 = 2$ with the first two rows being zero, and

rank $\mathscr{A}_1^2 =$ rank $\mathscr{A}_1^3 =$ rank $\mathscr{A}_1^2\mathscr{A}_0^2 =$ rank $\mathscr{A}_1\mathscr{A}_0\mathscr{A}_1^2 = 2$ with the last two rows being zero.

Hence the procedure of [7] stops and it follows that Φ is l. l. i.

Next we consider the problem whether a l. l. i. refinable function vector can have components with holes. The answer is positive and we present the following example.

Example 3. Let $\Phi = (\phi_1,\ \phi_2)^T$ be a nonzero compactly supported solution of the refinement equation

$$\Phi(t) = \begin{pmatrix} 1/9 & 2/3 \\ 1/9 & 1/3 \end{pmatrix}\Phi(2t) + \begin{pmatrix} 1/3 & 1 \\ 1/3 & 0 \end{pmatrix}\Phi(2t-1) + \begin{pmatrix} 2/3 & 0 \\ 1/9 & 0 \end{pmatrix}\Phi(2t-2)$$
$$+ \begin{pmatrix} 0 & 0 \\ 1/3 & 0 \end{pmatrix}\Phi(2t-7).$$

Then Φ is continuous and l. l. i. Moreover, supp $\phi_1 = [0,\ 3]$ and gsupp $\phi_2 = [0, 5]$ and ϕ_2 possesses a hole of length 1, namely $\phi_2(t) = 0$ for $t \in (5/2, 7/2)$, see Fig. 3.

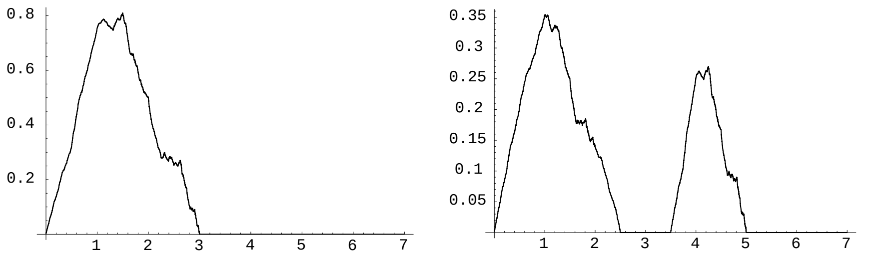

Fig. 3. Locally linearly independent $\Phi(t)$ where supp ϕ_2 possesses a hole.

Proof. We first prove continuity of Φ. The matrix $\frac{1}{2}\sum_{k=0}^{7} A(k)$ has the eigenvalues 1 and $-5/18$. Further, the mask satisfies the sum rule of order 1, namely, $(1,1)(A(0)+A(2))=(1,1)=(1,1)(A(1)+A(7))$. Hence, the (14×14)-matrices $\mathscr{A}_0$ and $\mathscr{A}_1$ both have the eigenvalue 1 with the corresponding left row eigenvector $e_1^T \coloneqq (1,1,\dots,1)$. Moreover, $\mathscr{A}_0$ and $\mathscr{A}_1$ are column-stochastic matrices, i.e., all entries in $\mathscr{A}_0$ and $\mathscr{A}_1$ are nonnegative and the sum of entries in each column is 1. Observe that a product of two column-stochastic matrices is again column-stochastic. A column-stochastic matrix is called *scrambling* if each pair of columns of A has positive entries in some common row. In particular, if A is column-stochastic and has a positive row, then A is scrambling.

Consider the subspace U of $\mathbb{R}^{14}$,

$$U \coloneqq \{u\in\mathbb{R}^{14} : e_1^T u = 0\}.$$

We apply the following result of Jia and Zhou [16] for stochastic matrices: A column-stochastic matrix is scrambling if and only if $\|A|_U\|<1$, where $\|\cdot\|$ denotes the maximum column sum norm of a matrix.

Hence continuity of Φ is already proved if we can find a $k\in\mathbb{N}$ such that for each k-tuple $(\varepsilon_1,\dots,\varepsilon_k)$, $\varepsilon_1,\dots,\varepsilon_k\in\{0,1\}$, the matrix product $\mathscr{A}_{\varepsilon_1}\dots\mathscr{A}_{\varepsilon_k}$ has a positive row (see [16, Theorem 1.1]).

A computation by Maple tells us for the matrix products $A_{\varepsilon_1}A_{\varepsilon_2}A_{\varepsilon_3}$ with $\varepsilon_j\in\{0,1\}$: If $(\varepsilon_1,\varepsilon_2)\neq(1,1)$, then the third and fourth rows of the matrix product are positive, while for $(\varepsilon_1,\varepsilon_2)=(1,1)$, even the first four rows of the matrix product are positive. This shows that all the column-stochastic matrices of the form $A_{\varepsilon_1}A_{\varepsilon_2}A_{\varepsilon_3}$ are scrambling. Hence the joint spectral radius $\rho(\mathscr{A}_0|_U,\mathscr{A}_1|_U)$ is less than 1. Therefore, the subdivision scheme associated with this mask converges uniformly, and Φ is continuous.

Let us now consider the space V, generated by an eigenvector of $\frac{1}{2}(\mathscr{A}_0+\mathscr{A}_1)$ to the eigenvalue 1,

$$v_0 = (6294, 50221/15, 12195, 12850/3, 4203, 689, 0, 1049, 0, 2733, 0, 0, 0, 0)^T.$$

Then V is spanned by the vectors v_0, $\mathscr{A}_0v_0$, $\mathscr{A}_0^2v_0$, $\mathscr{A}_1\mathscr{A}_0v_0$, $\mathscr{A}_0^3v_0$, $\mathscr{A}_1\mathscr{A}_0^2v_0$, $\mathscr{A}_0\mathscr{A}_1\mathscr{A}_0v_0$, $\mathscr{A}_1^2\mathscr{A}_0v_0$ and has dimension 8.

Let $\mathscr{B}$ be a (14×8)-matrix, such that the columns of $\mathscr{B}$ form a basis of V. Then $\mathscr{B}$ has 6 zero rows, namely the 7th, 9th, 11th, 12th, 13th and 14th row. Hence, by Theorem A, Φ is linearly independent on $(0,1)$. Since for continuous Φ, V contains $\mathbf{\Phi}(t) = (\Phi(t+k))_{k=0}^{6}$ for $t \in (0,1)$, it follows that $\operatorname{gsupp} \phi_1 = [0,3]$ and $\operatorname{gsupp} \phi_2 = [0,5]$.

We define the restricted vector $\tilde{\mathbf{\Phi}}(t)$ for $t \in [0,1)$ as

$$\tilde{\mathbf{\Phi}}(t) = (\phi_1(t), \phi_2(t), \phi_1(t+1), \phi_2(t+1), \phi_1(t+2),\\ \phi_2(t+2), \phi_2(t+3), \phi_2(t+4))^T.$$

Then $\tilde{V} = \operatorname{span}\{\tilde{\mathbf{\Phi}}(t) : t \in [0,1)\}$ has dimension 8.

Further, let us consider the matrices $\mathscr{B}_0, \mathscr{B}_1$, which are derived from $\mathscr{A}_0, \mathscr{A}_1$ by restricting to $\tilde{\mathbf{\Phi}}$, i.e., by deleting the 7th, 9th, 11th, 12th, 13th and 14th rows and columns of $\mathscr{A}_0, \mathscr{A}_1$,

$$9\mathscr{B}_0 = \begin{pmatrix} 1 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 3 & 9 & 1 & 6 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 9 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad 9\mathscr{B}_1 = \begin{pmatrix} 3 & 9 & 1 & 6 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 3 & 9 & 6 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that then (3) is of the form

$$\tilde{\mathbf{\Phi}}(t/2) = \mathscr{B}_0 \tilde{\mathbf{\Phi}}(t), \quad \tilde{\mathbf{\Phi}}((t+1)/2) = \mathscr{B}_1 \tilde{\mathbf{\Phi}}(t), \quad t \in [0,1).$$

Now we can choose $\mathscr{B}$ to be the 8×8 identity matrix and the procedure of Goodman et al. [7] (with $\mathscr{B}_0, \mathscr{B}_1$ instead of $\mathscr{A}_0, \mathscr{A}_1$) gives

$$\operatorname{rank} \mathscr{B}_0 = \operatorname{rank} \mathscr{B}_0^2 = \operatorname{rank} \mathscr{B}_0\mathscr{B}_1 = 7 \text{ and the 7th rows are zero,}$$

$$\operatorname{rank} \mathscr{B}_1 = \operatorname{rank} \mathscr{B}_1\mathscr{B}_0 = \operatorname{rank} \mathscr{B}_1\mathscr{B}_1 = 7 \text{ and the 6th rows are zero.}$$

Hence, Φ is l. l. i. Moreover, ϕ_2 possesses a hole of length 1, namely $\phi_2(t) = 0$ for $t \in (5/2, 7/2)$. $\square$

Remark. A similar example of a continuous l. l. i. function vector with one hole can be found in [20].

However, in certain cases one can show that the components of Φ cannot have holes.

Theorem 2. *Let $\Phi = (\phi_1, \dots, \phi_r)^T$ be a locally linearly independent vector of compactly supported L^1-functions satisfying*

$$\Phi(t) = \sum_{k=0}^{N} A(k)\Phi(2t-k)$$

for some $N \in \mathbb{N}$. Suppose that $A(0)$ and $A(N)$ contain no zero row. Then all nonzero components of Φ have support $[0, N]$, and in particular, they have no holes.

Proof. Deleting zero components of Φ, we may assume that each component ϕ_ν of Φ is nonzero.

We show first that $\operatorname{gsupp} \phi_\nu = [0, N]$, $\nu = 1, \dots, r$.

Let $\operatorname{gsupp} \phi_\nu = [a_\nu, b_\nu]$. We find from (3) that for each fixed μ,

$$\phi_\mu(t/2) = \sum_{\nu=1}^{r} A_{\mu,\nu}(0)\phi_\nu(t) + \sum_{k=1}^{N} \sum_{\nu=1}^{r} A_{\mu,\nu}(k)\phi_\nu(t-k).$$

Since $A(0)$ has no zero rows, at least one of the coefficients $A_{\mu,\nu}(0)$ is nonzero. On the interval $(-\infty, 2a_\mu), \phi_\mu(t/2)$ vanishes. By the local linear independence, for each ν with $A_{\mu,\nu}(0) \neq 0$ it follows that $\phi_\nu(t) = 0$ on this interval. Hence $a_\nu \geqslant 2a_\mu$, i.e., $a_\mu \leqslant \frac{1}{2} a_\nu$. Hence, by local linear independence, for all $\mu \in \{1, \dots, r\}$ there exists a ν with $2a_\mu - a_\nu = 0$.

Same arguments as in the proof of Theorem 1 imply that for all a_μ with μ in a cycle $\{\mu_1, \dots, \mu_d\}$, we have $a_{\mu_1} \leqslant \frac{1}{2} a_{\mu_2} \leqslant \cdots \leqslant \frac{1}{2^d} a_{\mu_d} \leqslant \frac{1}{2^{d+1}} a_{\mu_1}$. But $a_{\mu_1} \geqslant 0$. Then $a_{\mu_1} = \cdots = a_{\mu_d} = 0$. Further, for each $a_{\mu'}$ with μ' being not in a cycle, there exists a chain to a cycle and we find again $a_{\mu'} = 0$. Thus, for all components ϕ_ν of Φ, $\operatorname{gsupp} \phi_\nu$ starts at zero.

Analogously, using the assumption that $A(N)$ has no zero rows, it follows that $b_\nu = N$ for all $\nu = 1, \dots, r$.

Now, suppose that some components ϕ_ν of Φ have holes (i.e., intervals (a, b) with $0 < a < b < N$, where some ϕ_ν is identically zero). Then there exist holes with greatest length. Let us choose a hole with greatest length. Without loss of generality we suppose that ϕ_1 has such a hole (c, d) with $0 < c < d < N$.

Refinability of Φ implies that

$$\phi_1(t) = \sum_{\nu=1}^{r} \sum_{k=0}^{N} A_{1,\nu}(k)\phi_\nu(2t-k).$$

On the interval $(c, d), \phi_1(t) = 0$. By the local linear independence, for each (ν, k) with $A_{1,\nu}(k) \neq 0$ the corresponding function ϕ_ν needs to satisfy $\phi_\nu(t) = 0$ for $t \in (2c - k, 2d - k)$. But $\operatorname{gsupp} \phi_\nu = [0, N]$ and ϕ_ν does not have a hole with length $2(d-c)$. Hence either $2c - k \geqslant N$ or $2d - k \leqslant 0$.

If $c < N/2$, then the above discussion tells us that for each (v,k) with $A_{1,v}(k) \neq 0$, $2d - k \leqslant 0$. It follows that

$$\phi_1(t) = \sum_{v=1}^{r} \sum_{k \geqslant 2d} A_{1,v}(k)\phi_v(2t-k).$$

Then gsupp $\phi_1 \subset [d, N]$, contradicting the above observation that gsupp $\phi_1 = [0, N]$.

In the same way, if $c \geqslant N/2$, then $d > N/2$, and

$$\phi_1(t) = \sum_{v=1}^{r} \sum_{k \leqslant 2c-N} A_{1,v}(k)\phi_v(2t-k).$$

Then gsupp $\phi_1 \subset [0, c]$, which is again a contradiction. Therefore, ϕ_v cannot have holes. $\square$

Remark. We want to remark, that for g. l. i. function vectors Φ it has been shown by Ruch et al. [22] that $\cup_{v=1}^{r}$ supp $\phi_v = [0, N]$ if and only if A_0 and A_N are not nilpotent. Recently, Micchelli and Zhou [19] have studied the positivity of scalar refinable functions with nonnegative masks (inside the supports).

4. Bases of shift-invariant spaces

Let $\Phi \in (L^1(\mathbb{R}))^r$ be a vector of compactly supported functions ϕ_v, $v = 1, \ldots, r$. Denote by $S(\Phi)$ the linear space of all functions of the form

$$\sum_{v=1}^{r} \sum_{k \in \mathbb{Z}} c_v(k)\phi_v(\cdot - k) \tag{5}$$

with arbitrary sequences $c_v : \mathbb{Z} \to \mathbb{R}$. The space $S(\Phi)$ is a finitely generated shift-invariant space (FSI-space). The components ϕ_v of Φ are called generators of $S(\Phi)$. Further, let $S_0(\Phi)$ be the linear span of $\{\phi_v(\cdot - l) : v = 1, \ldots r, l \in \mathbb{Z}\}$, i.e., $S_0(\Phi)$ contains only finite linear combinations of $\phi_v(\cdot - l)$.

We want to deal with the following problem: Does an FSI-space $S(\Phi)$ possess a linearly independent basis over $(0, 1)$?

Our considerations are based on the following.

Theorem B. (Jia [12]). *Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported distributions on $\mathbb{R}$. Then there exists a distribution vector $\Psi = (\psi_1, \ldots, \psi_s)^T$ with the following properties*:

(a) *Ψ is globally linearly independent.*
(b) *$\Phi \subset S_0(\Psi)$, i.e., all components ϕ_v of Φ are finite linear combinations of integer translates of $\psi_1, \ldots, \psi_s$.*
(c) *$s \leqslant r$.*
(d) *$S(\Phi) = S(\Psi)$. Furthermore, if $\Phi \in (L^1(\mathbb{R}))^r$, then Ψ can be chosen with $\Psi \in (L^1(\mathbb{R}))^s$.*

In particular, each FSI-space $S(\Phi)$ possesses a globally linearly independent basis. This assertion is true even without assuming refinability of the vector Φ. Can we obtain a stronger result as formulated above? Unfortunately not, we obtain

Counterexample 1. Let $\Phi = (\phi_1, \phi_2)^T$ with ϕ_1 the normalized linear cardinal *B*-spline with support $[0, 2]$ (hat function) satisfying

$$\phi_1(t) = \frac{1}{2}\phi_1(2t) + \phi_1(2t-1) + \frac{1}{2}\phi_1(2t-2)$$

and with ϕ_2 satisfying the refinement equation

$$\phi_2(t) = \frac{1}{2}\phi_2(2t) + \frac{1}{2}\phi_1(2t-6) + \phi_1(2t-9).$$

Then Φ is a refinable vector of compactly supported, continuous functions and the FSI-space $S(\Phi)$ does not possess a linearly independent basis over $(0, 1)$, see Fig. 4.

Proof. We observe that $\operatorname{supp} \phi_2 \subset [0,\ 11/2]$. The (12×12)-matrices $\mathscr{A}_0$ and $\mathscr{A}_1$ can be simply derived from the refinement equations.

Let $\mathbf{\Phi}(t) := (\Phi(t)^T, \Phi(t+1)^T, \ldots, \Phi(t+5)^T)^T \in \mathbb{R}^{12}$. Then from (3) one obtains that the space $V = V_\Phi := \{\mathbf{\Phi}(t) : t \in [0, 1)\}$ is spanned by

$$\mathbf{\Phi}(0) = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1)^T,$$

$$\mathbf{\Phi}(1/2) = (1/2, 0, 1/2, 0, 0, 1/2, 0, 1/2, 0, 0, 0, 0)^T,$$

$$\mathbf{\Phi}(1/4) = (1/4, 0, 3/4, 1/4, 0, 0, 0, 1/4, 0, 0, 0, 1/2)^T,$$

$$\mathbf{\Phi}(3/4) = (3/4, 0, 1/4, 1/4, 0, 0, 0, 1/4, 0, 1/2, 0, 0)^T,$$

$$\mathbf{\Phi}(3/8) = (3/8, 0, 5/8, 0, 0, 1/4, 0, 3/8, 0, 0, 0, 1/4)^T,$$

$$\mathbf{\Phi}(5/8) = (5/8, 1/8, 3/8, 1/8, 0, 1/4, 0, 3/8, 0, 1/4, 0, 0)^T.$$

The orthogonal complement W of V is spanned by the unit vectors e_5, e_7, e_9, e_{11} and further by the vectors $w_1 = (0, 0, -1, 0, 0, 0, 0, 1, 0, 0, 0, 1)^T$ and $w_2 = (-1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0)^T$. Here, a unit vector e_j is defined by $e_j = (\delta_{j,k})_{k=1}^{12}$ with δ the Kronecker

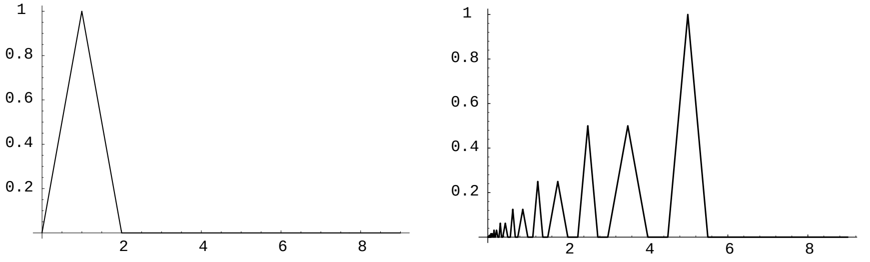

Fig. 4. $\Phi(t)$ generating no linearly independent basis over $(0, 1)$ of $S(\Phi)$.

symbol. The unit vectors in W are due to the support of ϕ_1 (being $[0,2]$ only). The vectors w_1, w_2 imply the local dependencies

$$-\phi_1(t) + \phi_2(t+3) + \phi_2(t+4) = 0,$$
$$-\phi_1(t+1) + \phi_2(t+3) + \phi_2(t+5) = 0$$

for $t \in (0,1)$, i.e., Φ is linearly dependent over (0, 1).

However, it can be simply observed that Φ is globally linearly independent, since W does not contain any vector of the form

$$(c_0, c_1, c_0\rho, c_1\rho, c_0\rho^2, c_1\rho^2, c_0\rho^3, c_1\rho^3, c_0\rho^4, c_1\rho^4, c_0\rho^5, c_1\rho^5)^T$$

with constants c_0, c_1, ρ (cf. [14, Theorem 3.3]).

We now prove, that there exists no basis of $S(\Phi)$ being linearly independent over $(0,1)$ by showing that the assumption that such a basis exists leads to a contradiction.

Suppose that there exists a refinable function vector $\Psi = (\psi_1, \psi_2)$ with $S(\Psi) = S(\Phi)$ and with Ψ being linearly independent over $(0,1)$. Then there exists a finite linear combination

$$\phi_1(t) = \sum_{k\in\mathbb{Z}} (a_k\psi_1(t-k) + b_k\psi_2(t-k)), \quad a_k, b_k \in \mathbb{R}.$$

Since $\operatorname{supp}\phi_1 = [0,2]$ it follows from the linear independence of Ψ over $(0,1)$ that $a_k = 0$ if $\operatorname{supp}\psi_1(\cdot - k) \nsubseteq [0,2]$ and $b_k = 0$ if $\operatorname{supp}\psi_2(\cdot - k) \nsubseteq [0,2]$. Hence, at least one of the functions ψ_1, ψ_2 has support contained in $[0,2]$. Let us suppose that $\operatorname{supp}\psi_1 \subseteq [0,2]$. Since $S(\Phi) = S(\Psi)$, it follows that V_Φ and V_Ψ have the same dimension 6, thus the length of $\operatorname{supp}\psi_2$ must be greater than 2 and we have

$$\phi_1(t) = \sum_{k\in\mathbb{Z}} a_k\psi_1(t-k).$$

Considering the Fourier transforms $\widehat{\Phi} = (\widehat{\phi}_1, \widehat{\phi}_2)^T$ and $\widehat{\Psi} = (\widehat{\psi}_1, \widehat{\psi}_2)^T$, we hence find

$$\widehat{\Phi}(u) = \begin{pmatrix} g_1(e^{-iu}) & 0 \\ g_3(e^{-iu}) & g_4(e^{-iu}) \end{pmatrix} \widehat{\Psi}(u)$$

with appropriate algebraic Laurent polynomials g_1, g_3, g_4. Since both Φ and Ψ are globally linearly independent, it follows that the transformation matrix is invertible for all $u \in \mathbb{C}$, and $g_1(z), g_4(z)$ have no zeros in $\mathbb{C}\backslash\{0\}$ (see [14]). But this is only true if $g_1(z) = z^{j_1}, g_4(z) = z^{j_2}$ for some integers j_1, j_2.

Without loss of generality, we can assume that $\operatorname{supp}\psi_1$ and $\operatorname{supp}\psi_2$ start in [0, 1). Hence $\phi_1(t) = a_0\psi_1(t)$, i.e., ψ_1 is the hat function. Further, according to $g_4(z) = z^{j_2}$ and to the assumed linear independence of Ψ, ϕ_2 satisfies

$$\phi_2(t) = \sum_{k=0}^{4} c_k\psi_1(t-k) + d_{j_2}\psi_2(t-j_2), \quad c_k,\ d_{j_2} \in \mathbb{R}$$

such that $\operatorname{supp} \psi_2(\cdot - j_2) \subseteq [0, 6]$. Now, the structure of ϕ_2 implies that $j_2 = 0$, since $\phi_2(t)$, $t \in [0, 1)$, cannot be represented by the hat function $\psi_1(t)$. Hence the dependence relation $-\phi_1(t) + \phi_2(t+3) + \phi_2(t+4) = 0$ for $t \in (0, 1)$ causes

$$\begin{aligned} &- a_0\psi_1(t) + (d_0\psi_2(t+3) + c_2\psi_1(t+1) + c_3\psi_1(t)) \\ &\quad + (d_0\psi_2(t+4) + c_3\psi_1(t+1) + c_4\psi_1(t)) = 0 \end{aligned}$$

for $t \in (0, 1)$, i.e., Ψ is not linearly independent over $(0, 1)$ and we have found the desired contradiction. □

Finally, let us consider the following question: Let Φ be a refinable vector of compactly supported L^1-functions. If Φ is globally linearly independent, is there a finite interval (t_1, t_2), $t_1, t_2 \in \mathbb{R}$, such that Φ is linearly independent over (t_1, t_2)?

We find the following.

Counterexample 2. Let $\Phi = (\phi_1, \phi_2)^T$ with $\phi_1(t) = \chi_{[0,1)}(t)$, where $\chi_{[0,1)}$ denotes the characteristic function on $[0, 1)$, and with

$$\phi_2(t) = \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} (\chi_{[0,1)}(2^j t - 2) + \chi_{[0,1)}(2^j t - 3)).$$

Then Φ is a refinable, globally linearly independent vector of compactly supported L^1-functions being linearly dependent over any finite interval (t_1, t_2), $t_1, t_2 \in \mathbb{R}$, see Fig. 5.

Proof. The vector Φ is refinable with

$$\begin{aligned} \phi_1(t) &= \phi_1(2t) + \phi_1(2t-1), \\ \phi_2(t) &= \frac{1}{2}\phi_2(2t) + \phi_1(2t-2) + \phi_1(2t-3). \end{aligned}$$

Further, we find $\operatorname{supp} \phi_1 = [0, 1]$ and $\operatorname{supp} \phi_2 = [0, 2]$.

We first show that Φ is globally linearly independent. Let a, b be sequences such that

$$\sum_{k \in \mathbb{Z}} (a(k)\phi_1(t-k) + b(k)\phi_2(t-k)) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{6}$$

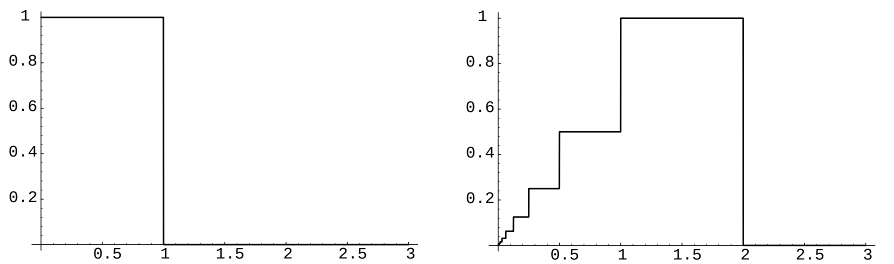

Fig. 5. $\Phi(t)$ being g. l. i. but not linearly dependent on any interval (t_1, t_2).

Suppose first that one component of b is nonzero, say $b(l) \neq 0$ for a fixed $l \in \mathbb{Z}$. Considering (6) for $t \in (l, l+1)$, we obtain (according to the support of ϕ_1, ϕ_2)

$$a(l)\phi_1(t-l) + b(l)\phi_2(t-l) + b(l-1)\phi_2(t-l+1) = 0$$

and by definition of ϕ_1 and ϕ_2 hence

$$a(l) + b(l)\phi_2(t-l) + b(l-1) = 0,$$

since $\phi_1(t-l)$ and $\phi_2(t-l+1)$ are identically 1 for $t \in (l,\ l+1)$. However, $\phi_2(t-l)$ can take all values $1/2^n$, $n = 1, 2, \ldots$. Hence the above equation can only be satisfied if $b(l) = 0$, contradicting our assumption. Thus, b is a zero sequence. But now, (6) simply implies that also a must be a zero sequence, i.e., Φ is globally linearly independent.

However, Φ is linearly dependent over every finite interval $(t_1, t_2) \subset \mathbb{R}$, since we find

$$-\phi_1(t - N_1) + \phi_2(t - N_1 + 1) = 0 \quad \text{for all } t \in (t_1, t_2),$$

where N_1 is the greatest integer less than t_1. $\square$

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